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Analysis on Transient Heat Transfer in Straight Fins of Various Shapes with Its Base Subjected to a Constant Heat Flux

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The theoretical study of the heat flow within finned heat exchangers is of considerable practical importance because of the extensive utilization of a finned surface for heat transfer enchancement in applications varying from air-cooled heat exchangers in the process industries to heat rejection equipment in space vehicles. This paper investigates the transient heat transfer in two-dimensional straight fins of various shapes with the base subjected to a constant heat flux. In order to obtain the solutions of the governing equation which is a partial differential equation, the following procedures of analysis should be performed:

(1) Normalize the governing partial differential equation subjected to the appropriate initial and boundary conditions.

(2) Take the Laplace transform with respect to time.

(3) Utilize the integral method to solve the transformed system.

(4) Achieve the inverse Laplace transform by the Fourier series technique.

INTRODUCTION

The theoretical study of the heat flow within finned heat exchangers is of considerable practical importance because of the extensive utilization of a finned surface for heat transfer enchancement in applications varying from air-cooled heat exchangers in the process industries to heat rejection equipment in space vehicles.

Heat transfer phenomena in extended surfaces of various shapes made of a single material have been investigated and published in the works of, among others, Harper and Brown [1], Jacob [2], Gardner [3], Carier and Anderson [4], and Gates [5].

The early investigations into the applicability of the one-dimensional

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approximation restricted attention solely to the fin and concluded that two-dimensional effects are negligible provided the transverse Biot number, based on the fin-base thickness, is much less than unit [6–8]. Recent investigations of the combined fin and supporting surface have shown that the presence of fins induces two-dimensional effects within the supporting surface and these may in turn act to produce two-dimensional variations within the fin, e.g., [9–12]. Suryanarayana [9] has reported that the difference between heat transfer rates predicted by one- and two-dimensional analysis can be as much as 80%. It is therefore essential for the effective design of finned heat exchangers to consider the complete fin assembly and to employ a multi-dimensional analysis.

In Suryanarayana [13, 14], the base of a straight fin is subjected to a step change in temperature, heat flux, or fluid temperature, and the forms in sinusoidal temperature, heat flux, and fluid temperature with unit amplitude are also analyzed. In his paper, the solutions of transient response of straight fins are evaluated through the method of Laplace transforms which enables us to easily evaluate rapidly convergent approximate solutions for small values of time. Chu [15–17] determined the transient response of two-dimensional straight fins, two-dimensional circular pins, and one-dimensional annular and composite straight fins by using the Fourier series inversion technique.

This paper investigates the transient heat transfer in two-dimensional straight fins of various shapes with the base subjected to a constant heat flux. In order to obtain the solutions of the governing equation which is a partial differential equation, the following procedures of analysis should be performed:

(1) Normalize the governing partial differential equation subjected to the appropriate initial and boundary conditions.

(2) Take the Laplace transform with respect to time.

(3) Utilize the integral method [18–20] to solve the transformed system.

(4) Achieve the inverse Laplace transform by the Fourier series technique [15–17].

ANALYSIS

Consider the two-dimensional straight fins of various shapes as shown in Fig. 1. The material of the fin is isotropic and homogeneous, and all the thermal properties are assumed to be constant. The fin tip is also taken to be insulated which is equivalent to neglecting the fin tip heat transfer due to its small area. The ambient temperature is constant and no heat sources

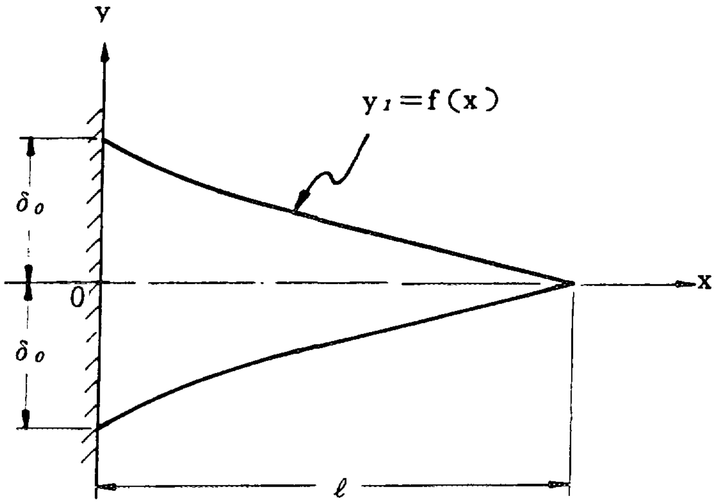

FIG. 1. The configuration of the straight fin.

or sinks are present in the fin. The base of the fin is subjected to a constant heat flux. Since the geometry and the boundary conditions are symmetric about the centerline of the fin, only the analysis for the upper half region is necessary.

The differential equation to be solved is

$$\rho c \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \qquad 0 \leqslant x \leqslant l, 0 \leqslant y \leqslant f(x), t > 0. \tag{1}$$

The initial and boundary conditions are

$$T(x, y, 0) = T_\infty, \qquad 0 \leqslant x \leqslant l, 0 \leqslant y \leqslant f(x) \tag{2}$$

$$q_b = -k \left.\frac{\partial T}{\partial x}\right|_{x=0}, \qquad 0 \leqslant y \leqslant \delta_0, t > 0 \tag{3}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=l} = 0, \qquad 0 \leqslant y \leqslant f(x), t > 0 \tag{4}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \qquad 0 \leqslant x \leqslant l, t > 0 \tag{5}$$

$$k \left.\frac{\partial T}{\partial n}\right|_{y=f(x)} + h[T(x, f(x), t) - T_\infty] = 0$$

or

$$\left.\frac{k(\partial T/\partial y - y'(\partial T/\partial x))}{[1+(y')^2]^{1/2}}\right|_{y=f(x)} + h[T(x, f(x), t) - T_\infty] = 0,$$
$$0 \leqslant x \leqslant l,\ t > 0. \tag{6}$$

Introducing the following dimensionless quantities θ, X, Y, τ, Bi, A, and Z as defined in the nomenclature, Eqs. (1)–(6) become

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}, \qquad 0 \leqslant X \leqslant 1, 0 \leqslant Y \leqslant Z, \tau > 0 \tag{7}$$

$$\theta(X, Y, 0) = 0, \qquad 0 \leqslant X \leqslant 1, 0 \leqslant Y \leqslant Z \tag{8}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{x=0} = -1, \qquad 0 \leqslant Y \leqslant A, \tau > 0 \tag{9}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{x=1} = 0, \qquad 0 \leqslant Y \leqslant Z, \tau > 0 \tag{10}$$

$$\left.\frac{\partial \theta}{\partial Y}\right|_{y=0} = 0, \qquad 0 \leqslant X \leqslant 1, \tau > 0 \tag{11}$$

$$\left.\frac{\partial \theta/\partial Y - Z'(\partial \theta/\partial X)}{[1+(Z')^2]^{1/2}}\right|_{y=z} + Bi\theta(X, Y, \tau) = 0, \qquad 0 \leqslant X \leqslant 1, \tau > 0. \tag{12}$$

Equations (7)–(12) represent a typical initial and boundary value problem. We first take the Laplace transform with respect to time, and the system then becomes a boundary value problem:

$$S\tilde{\theta} = \frac{\partial^2 \tilde{\theta}}{\partial X^2} + \frac{\partial^2 \tilde{\theta}}{\partial Y^2}, \qquad 0 \leqslant X \leqslant 1, 0 \leqslant Y \leqslant Z \tag{13}$$

$$\frac{\partial \tilde{\theta}}{\partial X}(0, Y) = -\frac{1}{S}, \qquad 0 \leqslant Y \leqslant A \tag{14}$$

$$\frac{\partial \tilde{\theta}}{\partial X}(1, Y) = 0, \qquad 0 \leqslant Y \leqslant Z \tag{15}$$

$$\frac{\partial \tilde{\theta}}{\partial Y}(X, 0) = 0, \qquad 0 \leqslant X \leqslant 1 \tag{16}$$

$$\left.\frac{\partial \tilde{\theta}/\partial Y - Z'(\partial \tilde{\theta}/\partial X)}{[1+(Z')^2]^{1/2}}\right|_{y=z} + Bi\tilde{\theta}(X, Z) = 0, \qquad 0 \leqslant X \leqslant 1. \tag{17}$$

We can then expand the dimensionless temperature by the integral method in the following way:

$$\tilde{\theta}(X, Y) = \alpha(Y) + \beta(Y)X + \gamma(Y)X^2 \tag{18}$$

Substituting Eq. (18) into Eqs. (14) and (15), we obtain

$$\beta(Y) = -\frac{1}{S} \quad \text{and} \quad \gamma(Y) = \frac{1}{2S}. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (13), and integrating with respect to X from $X=0$ to $X=1$, then

$$\frac{d^2\alpha(Y)}{dY^2} - S\alpha(Y) = -\frac{1}{S} - \frac{1}{3}, \tag{20}$$

where

$$\alpha(Y) = C_1 e^{\sqrt{S}\,Y} + C_2 e^{-\sqrt{S}\,Y} + \frac{1}{S^2} + \frac{1}{3S}. \tag{21}$$

Applying boundary conditions, i.e., Eqs. (16) and (17), the solution of Eq. (13) is then

$$\tilde{\theta}(X, Y) = C_1(e^{\sqrt{S}\,Y} + e^{-\sqrt{S}\,Y}) - \frac{X}{S} + \frac{X^2}{2S} + \frac{1}{S^2} + \frac{1}{3S}, \tag{22}$$

where

$$\begin{aligned} C_1 &\left\{\frac{\sqrt{S}\,(e^{\sqrt{S}\,Z} - e^{-\sqrt{S}\,Z})}{[1+(Z')^2]^{1/2}} + Bi(e^{\sqrt{S}\,Z} + e^{-\sqrt{S}\,Z})\right\} \\ &= \left\{-\frac{Z'}{S}\frac{(1-X)}{[1+(Z')^2]^{1/2}} + Bi\left(\frac{X}{S} - \frac{X^2}{2S} - \frac{1}{S^2} - \frac{1}{3S}\right)\right\}. \end{aligned} \tag{23}$$

Results

(i) *Case A.* For the concave parabolic profile

$$f(x) = \delta_0\left(1 - \frac{x}{l}\right)^2, \qquad \text{i.e.,} \quad Z = A(1-X)^2. \tag{24}$$

Substituting Eq. (24) into (23), and integrating with respect to $(1-X)^2$ from $X=0$ to $X=1$, then (Appendix)

$$C_1 = \frac{[(1-2A^2)\sqrt{1+4A^2}-1]/6SA^3 + Bi(S+12)/12S^2}{\left(\begin{array}{l} (Bi/\sqrt{S}\,A)(e^{-\sqrt{S}A} - e^{\sqrt{S}A}) \\ \qquad -2SA(1/2 + SA^2/24 + S^2A^4/720 - 2A^2/3 - SA^4/15 + 3A^4/2) \end{array} \right)}. \tag{25}$$

(ii) *Case* B. For the triangle profile

$$f(x) = \delta_0 \left(1 - \frac{x}{l}\right), \qquad \text{i.e.,} \quad Z = A(1-X). \tag{26}$$

Substituting Eq. (26) into (23), and integrating with respect to $(1-X)$ from $X=0$ to $X=1$, then

$$C_1 = \frac{A/2S\sqrt{1+A^2} - Bi/S^2}{(e^{\sqrt{S}A} + e^{-\sqrt{S}A} - 2)/A\sqrt{1+A^2} + Bi(e^{\sqrt{S}A} - e^{-\sqrt{S}A})/\sqrt{S}\,A}. \tag{27}$$

(iii) *Case* C. For the rectangular profile

$$f(x) = \delta_0, \qquad \text{i.e.,} \quad Z = A. \tag{28}$$

Substituting Eq. (28) into (23), and integrating with respect to X from $X=0$ to $X=1$, then

$$C_1 = \frac{-Bi/S^2}{\sqrt{S}\,(e^{\sqrt{S}A} - e^{-\sqrt{S}A}) + Bi(e^{\sqrt{S}A} + e^{-\sqrt{S}A})}. \tag{29}$$

The function $\tilde{\theta}$ is very complex and it is difficult to find the inverse Laplace transform by the residue theorem. In this paper, a general method, known as the Fourier series technique [15–17], is used to obtain the inverse Laplace transform by the following numerical inversion formula:

$$\theta(T^*) \cong \frac{e^{CT^*}}{T^*} \left\{ \frac{1}{2}\tilde{\theta}(C) + \mathrm{Re} \sum_{k=1}^{N} \tilde{\theta}(C + ik\pi/T^*) \cdot (-1)^k \right\}, \tag{30}$$

from which the numerical results at the point, T^*, can be calculated by computer.

Discussion

Typical examples for the above three cases are shown in Figs. 2–11, which are typical dimensionless plots of the temperature distribution of the

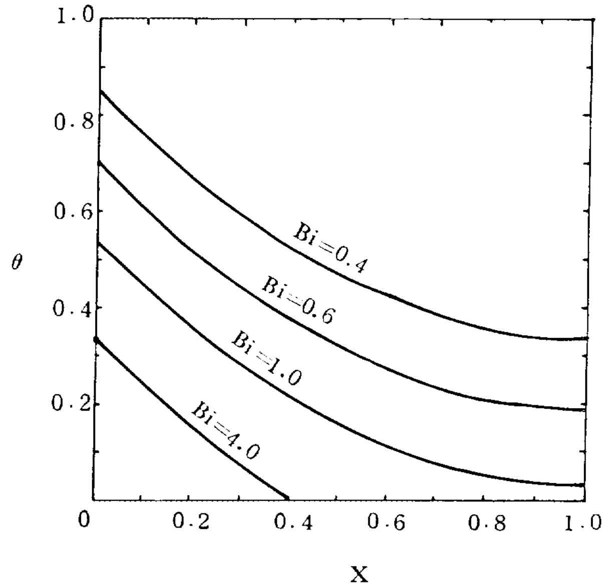

FIG. 2. The dimensionless temperature distribution θ versus X along the center line of the fin for concave parabolic profile at $\tau = 0.4$ and $A = 0.2$.

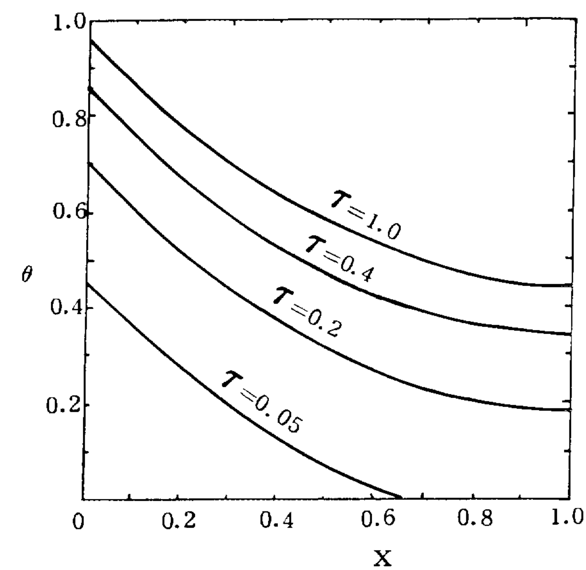

FIG. 3. The dimensionless temperature distribution θ versus X along the center line of the fin for concave parabolic profile at $Bi = 0.4$ and $A = 0.2$.

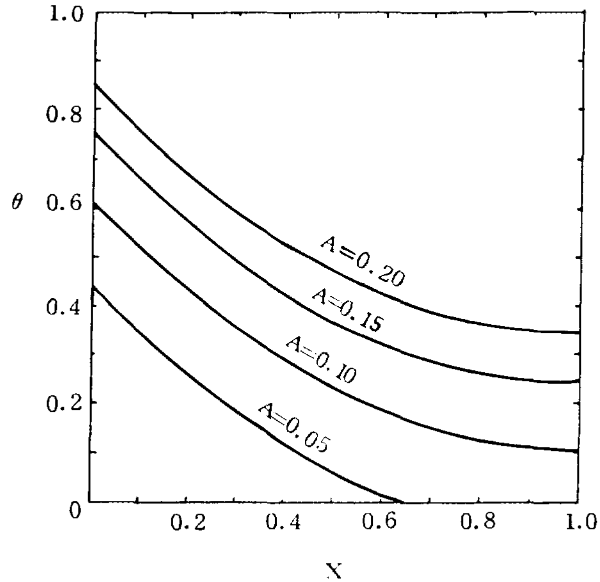

FIG. 4. The dimensionless temperature distribution θ versus X along the center line of the fin for concave parabolic profile at $Bi = 0.4$ and $\tau = 0.4$.

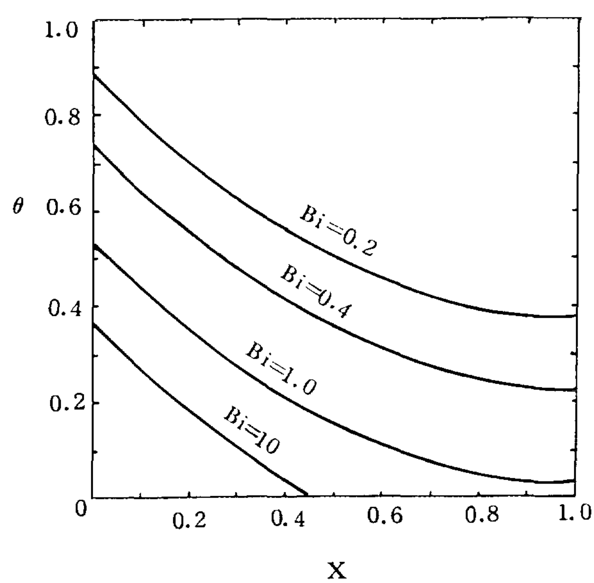

FIG. 5. The dimensionless temperature distribution θ versus X along the center line of the fin for triangle profile at $\tau = 0.4$ and $A = 0.2$.

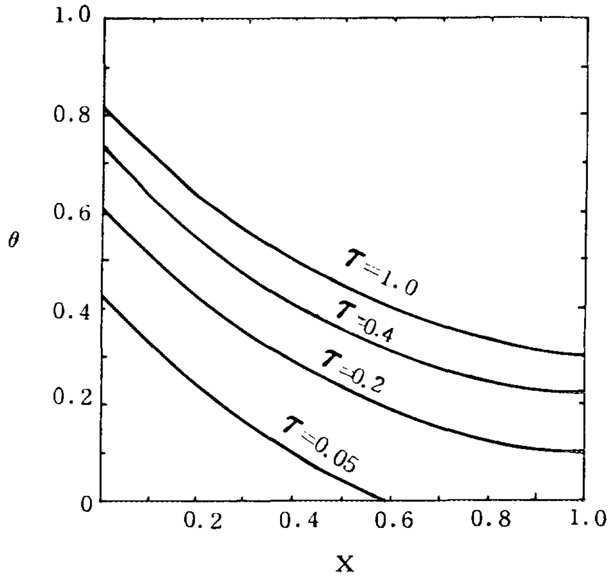

FIG. 6. The dimensionless temperature distribution θ versus X along the center line of the fin for triangle profile at $Bi = 0.4$ and $A = 0.2$.

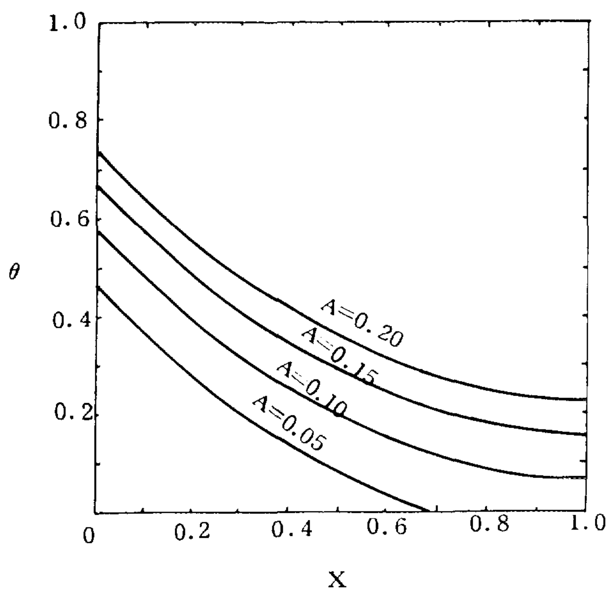

FIG. 7. The dimensionless temperature distribution θ versus X along the center line of the fin for triangle profile at $Bi = 0.4$ and $\tau = 0.4$.

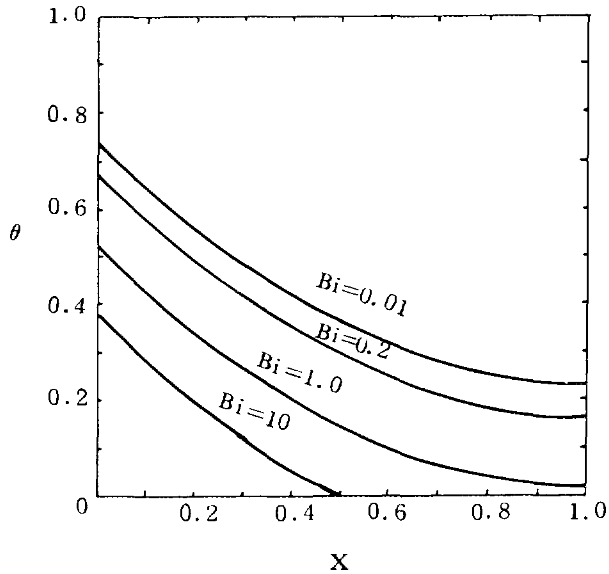

FIG. 8. The dimensionless temperature distribution θ versus X along the center line of the fin for rectangular profile at $\tau = 0.4$ and $A = 0.2$.

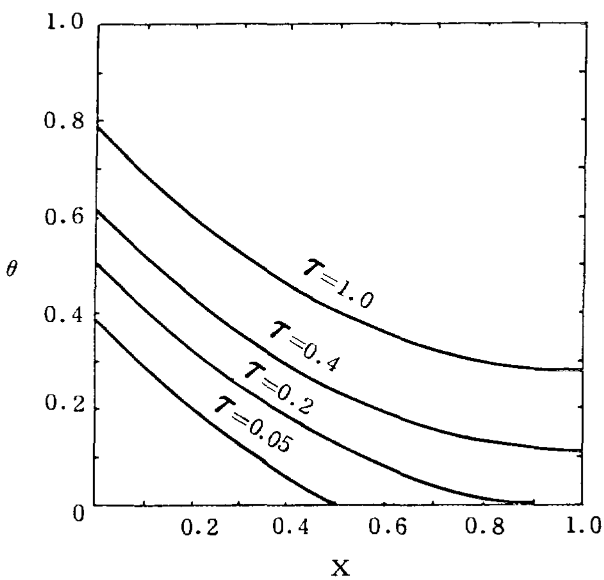

FIG. 9. The dimensionless temperature distribution θ versus X along the center line of the fin for rectangular profile at $Bi = 0.4$ and $A = 0.2$.

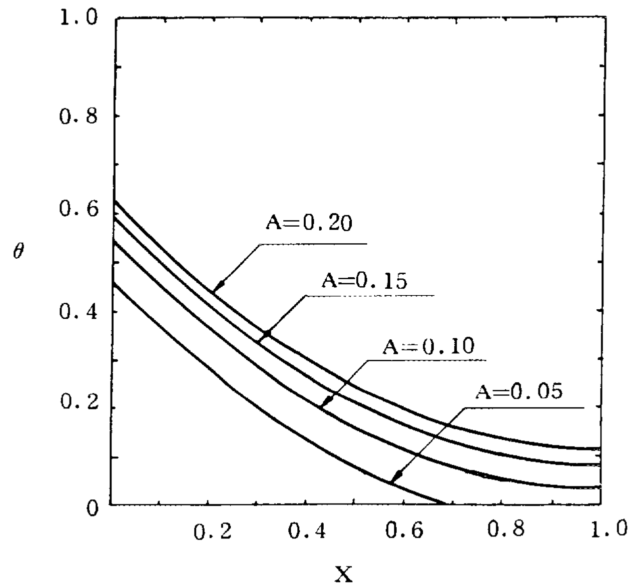

FIG. 10. The dimensionless temperature distribution θ versus X along the center line of the fin for rectangular profile at $Bi = 0.4$ and $\tau = 0.4$.

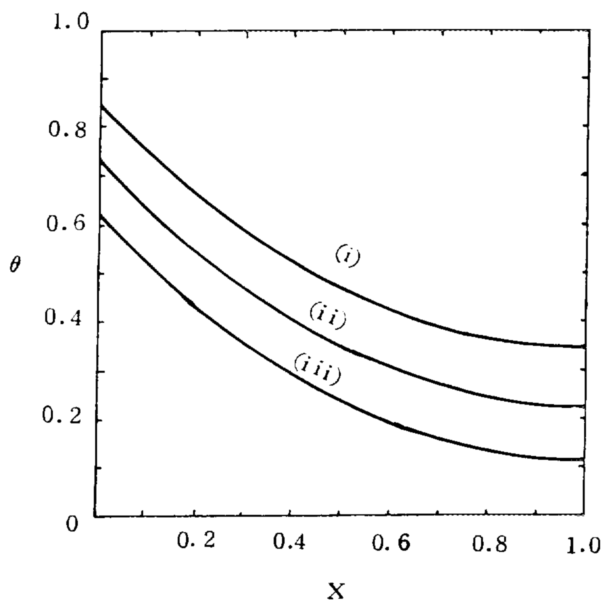

FIG. 11. The dimensionless temperature distribution θ versus X along the center line of the fin for various shapes at $Bi = 0.4$ and $A = 0.2$: (i) for concave parabolic profile; (ii) for triangle profile; (iii) for rectangular profile.

fin at the center line θ versus X along the center line for different values of τ, A, and Bi. The above results shown in these plots are the transient response in straight fins of various shapes with the base subjected to a constant heat flux.

It is shown that:

(1) The temperature distribution of the center line increases as the time increases, and comes to reach the steady state.

(2) The temperature distribution of the center line decreases as the distance away from the base of the fin increases.

(3) The temperature distribution of the center line decreases as the Biot number Bi increases. It is reasonable that, because the large value of Bi means that the heat transfer coefficient must be high, the heat convection from the fin surface to the surroundings must be high, so this causes less heat to be stored in the fin material, and the temperature distribution must decrease.

(4) The earlier the steady state can be reached the Biot number Bi increases, because the large value of Bi causes the temperature distribution along the center line away from the base of fin to decrease rapidly. So the steady state can be reached earlier.

(5) The temperature distribution of the center line increases as the half dimensionless thickness A increases, because the heat conduction increases as the value of A increases, and the earlier the steady state can be reached.

(6) The temperature distribution of the fin for the triangle profile is lower than that for the concave parabolic profile, but higher than that for the rectangular profile, because the smaller cross-sectional area causes higher heat conduction through the fin material.

APPENDIX

Nomenclature

A	dimensionless half thickness of the base of the fin, $A = \delta_0/l$
Bi	Biot number, $Bi = hl/k$
c	specific heat of material of the fin
$f(x)$	specific profile of the fin
h	heat transfer coefficient
k	thermal conductivity
l	length of the fin
n	normal direction of the surface of the fin
q_b	heat flux from the base of the fin
t	time

T	temperature of the fin
T_∞	ambient temperature
x, y	rectangular coordinates
X, Y	dimensionless rectangular coordinates, $X = x/l$, $Y = y/l$
Z	dimensionless specific profile of the fin, $Z = f(x)/l$
δ_0	half thickness of the base of the fin
ρ	density of material of the fin
τ	dimensionless time, $\tau = kt/\rho cl^2$
θ	dimensionless temperature of the fin, $\theta = k(T - T_\infty)/q_b l$

Substituting Eq. (24) into (23), and integrating with respect to $(1\text{-}X)^2$ from $X=0$ to $X=1$, then

$$C_1 \int_0^1 \left\{ \frac{\sqrt{S}\,[e^{\sqrt{S}A(1-X)^2} - e^{-\sqrt{S}A(1-X)^2}]}{\sqrt{1+4A^2(1-X)^2}} + Bi[e^{\sqrt{S}A(1-X)^2} + e^{-\sqrt{S}A(1-X)^2}] \right\} d(1-X)^2$$

$$= \int_0^1 \left[\frac{2A(1-X)(1-X)}{S\sqrt{1+4A^2(1-X)^2}} + Bi\left(\frac{X}{S} - \frac{X^2}{2S} - \frac{1}{S^2} - \frac{1}{3S}\right)\right] d(1-X)^2$$

$$= \int_0^1 \left\{ \frac{2A}{S}\frac{(1-X)^2}{\sqrt{1+4A^2(1-X)^2}} + Bi\left[\frac{1-(1-X)^2}{2S} - \frac{(S+3)}{3S^2}\right]\right\} d(1-X)^2. \tag{31}$$

From the table of integrals, i.e.,

$$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

The first term of integral in the right hand of Eq. (31) becomes

$$\frac{2A}{S}\left[-\frac{2\{2-4A^2(1-X)^2\}}{3(4A^2)^2}\sqrt{1+4A^2(1-X)^2}\right]_0^1$$

$$= \frac{(1-2A^2)\sqrt{1+4A^2}-1}{6SA^3}. \tag{32}$$

The second term of integral in the right hand of Eq. (31) becomes

$$Bi\left[\frac{(1-X)^2-(1-X)^4/2}{2S} - \left(\frac{S+3}{3S^2}\right)(1-X)^2\right]_0^1$$

$$= Bi\left[0 - \left(\frac{1-1/2}{2S}\right) - \left(-\frac{S+3}{2S^2}\right)\right] = Bi\frac{(S+12)}{12S^2}. \tag{33}$$

From the series expansion, i.e.,

$$\frac{1}{2}(e^{x}-e^{-x})=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots,\qquad x^2<1 \tag{34}$$

$$(1+x)^{-1/2}=1-\frac{x}{2}+\frac{1}{2}\cdot\frac{3}{4}x^2-\cdots,\qquad x^2<1. \tag{35}$$

And let

$$u=(1-X)^2,\ \int_0^1\cdots d(1-X)^2=\int_1^0\cdots du. \tag{36}$$

The first term of the integral in the left hand of Eq. (31) becomes

$$\begin{aligned}
&\int_1^0 \frac{\sqrt{S}\,(e^{\sqrt{S}Au}-e^{-\sqrt{S}Au})\,du}{\sqrt{1+4A^2u}}\\
&=\int_1^0 2\left[\sqrt{S}\,(\sqrt{S}\,Au)+\frac{(\sqrt{S}\,Au)^3}{3!}+\frac{(\sqrt{S}\,Au)^5}{5!}+\cdots\right]\\
&\quad\cdot\left[1-\frac{4A^2u}{2}+\frac{3}{8}(4A^2u)^2\cdots\right]du\\
&=\int_1^0 2SA\left[u+\frac{SA^2u^3}{6}+\frac{S^2A^4u^5}{120}-2A^2u^2-\frac{SA^4u^4}{3}+6A^4u^3\right]du\\
&=2SA\left[\frac{u^2}{2}+\frac{SA^2u^4}{24}+\frac{S^2A^4u^6}{720}-\frac{2A^2u^3}{3}-\frac{SA^4u^5}{15}+\frac{6A^4u^4}{4}\right]_1^0\\
&=-2SA\left(\frac{1}{2}+\frac{SA^2}{24}+\frac{S^2A^4}{720}-\frac{2A^2}{3}-\frac{SA^4}{15}+\frac{3A^4}{2}\right).
\end{aligned} \tag{37}$$

From Eq. (36), the second term of integral in the left hand of Eq. (31) becomes

$$\begin{aligned}
&\int_1^0 \frac{Bi}{\sqrt{S}\,A}(e^{\sqrt{S}Au}+e^{-\sqrt{S}Au})\,d(\sqrt{S}\,Au)\\
&=\frac{Bi}{\sqrt{S}\,A}\left[e^{\sqrt{S}Au}-e^{-\sqrt{S}Au}\right]_1^0\\
&=\frac{Bi}{\sqrt{S}\,A}(e^{-\sqrt{S}A}-e^{\sqrt{S}A}).
\end{aligned} \tag{38}$$

Substituting Eqs. (32), (33), (37), and (38) into Eq. (31), we obtain

$$C_1 = \frac{[(1-2A^2)\sqrt{1+4A^2}-1]/6SA^3 + Bi(S+12)/12S^2}{\left(\begin{array}{l}(Bi/\sqrt{S}\,A)(e^{-\sqrt{S}\,A}-e^{\sqrt{S}\,A}) \\ \quad -2SA(1/2+SA^2/24+S^2A^4/720-2A^2/3-SA^4/15+3A^4/2)\end{array}\right)}. \tag{25}$$

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